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Target-space anomalies and elliptic indices in heterotic orbifolds

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Abstract: We present a complete string theory analysis of all mixed gauge, gravitational and target-space anomalies potentially arising in the simplest heterotic $\mathbb{Z}_N$ orbifold models, with N odd and standard embedding. These anomalies turn out to be encoded in an elliptic index, which can be easily computed; they are found to cancel through a universal GS mechanism induced by the dilaton multiplet. The target-space symmetry is then shown to have a nice geometric interpretation in terms of torsion, and the target-space dependence of the four-dimensional GS couplings can be alternatively rederived from the implicit torsion dependence of the standard ten-dimensional GS couplings. The result is universal and consists essentially of a Bianchi identity for the NSNS B field depending on all the curvatures, and in particular on the target-space curvature.

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1. Introduction

Target-space duality symmetries (see [1] and references therein) in orbifold compactifications of heterotic string theory [2] have received a lot of attention in the past and have recently been the object of renewed interest [3]–[7] in the context of heterotic-type-I duality [8], which has been conjectured to exist also between certain $D = 4$ heterotic and type-IIB orientifold vacua [9]–[13]. At the level of the low-energy supergravity effective action, these symmetries correspond typically to global isometries acting on the manifold describing the moduli space of scalar fields. An example is given by $D = 4$ $N = 1$ heterotic orbifold models, which admit $\mathrm{SL}(2,\mathbb{R})_i$ target-space duality symmetries acting in a peculiar way on the untwisted moduli T_i that describe the Kähler structure of the internal space [14]. Such symmetries are

in general affected by one-loop anomalies[1] [17]–[21], but since a discrete version of them (T-duality) is believed to be an exact symmetry of heterotic string theory, the latter are expected to cancel by some mechanism.

Although anomalies in target-space duality symmetries (in the following denoted simply target-space anomalies) can arise in different channels, most of the attention in the past has been devoted to mixed target-space/gauge and target-space/gravitational anomalies for their relation to threshold corrections. It was realized that such field theory one-loop anomalies can be generically cancelled by the combination of a universal Green-Schwarz (GS) mechanism [22]–[24] involving the dilaton multiplet and the anomalous variation of threshold corrections to gauge couplings. The presence of a GS coupling has also been confirmed in [25] by a string computation. Other cases, such as purely target-space anomalies, have received little attention to date, and as far as we know, have been analysed only in [21], from a low-energy field theory point of view. It should be noticed that in all previous analyses, target-space anomalies have always been computed indirectly, by exploiting their similarity with gauge anomalies. The difficulty of a direct computation resides in the fact that the connection associated to a target-space duality symmetry is a composite rather than an elementary field.

In this paper, we reconsider this issue through an explicit string theory computation of all possible target-space/gauge/gravitational anomalies in heterotic orbifold models. We restrict to models without threshold corrections, where anomalies are expected to be cancelled through a GS mechanism only, and focus on the simplest $\mathbb{Z}_N$ models with N odd and standard embedding, i.e. the $\mathbb{Z}_3$ and the $\mathbb{Z}_7$ models. Anomalous amplitudes in heterotic string theory have been extensively studied in the past. In this work, we proceed along the lines of [26]–[28] and [29]–[32] to identify the generalizations of the elliptic genus [33] that are relevant to the anomalies under consideration. By evaluating both the modular-invariant and holomorphic versions of the elliptic genus, we are then able to derive both the one-loop field theory anomaly and the GS couplings. We show that the one-loop anomaly nicely factorizes and is completely cancelled by a GS mechanism. In a low-energy description in terms of chiral multiplets, the latter is induced by an inhomogeneous transformation of the dilaton multiplet, and the cancellation of pure target-space anomalies requires an appropriate kinetic term for the composite target-space connection in the low-energy effective action, as already conjectured in [21]. In a dual description in terms of linear multiplets, this term modifies the Bianchi identity for the NSNS two-form B dual to the universal axion by a source term involving the target-space curvature.

The results we find agree with those derived in [19, 20] for target-space/gauge and target-space/gravitational anomalies, and with the various modular weights assign-

[1] The presence of anomalies in this kind of symmetries was first addressed in [15] and their possible cancellation through a generalized GS mechanism noted in [16].

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ments of [20]. However, a disagreement with the results of [21] for pure target-space anomalies is found. This suggests that the analogy between target-space and gauge anomalies used in [21] could present subtleties related to the compositeness of the target-space connection, and might be incorrect when more than one composite connection occurs as external states. In this respect, we have checked that the anomaly computed along the lines of [21] does not factorize and hence could not be cancelled by a GS mechanism.

We also clarify the geometrical structure of target-space anomalies by showing that the four-dimensional target-space curvatures are nothing but the internal components of the ten-dimensional torsion-full curvature two-form. The additional term in the Bianchi identity for B can then be deduced from the implicit torsion dependence of the ten-dimensional GS term, pointed out in [34]. A nice geometric interpretation can also be made of the contribution of each single state to the anomaly in our string set-up. In particular, target-space anomalies resemble standard gravitational and gauge anomalies in the untwisted and twisted sectors of the orbifold, respectively. This is explained by the quite different dependence on the internal metric and volume in the two cases.

The structure of this paper is as follows. In section 2, we review well-known general properties of anomalous heterotic amplitudes. In section 3, we describe our strategy for the string computation and hence compute the elliptic genus relevant to all target-space/gauge/gravitational anomalies. Some comments on the field-theory interpretation of our results are then given in section 4, whereas the geometric structure of these anomalies, including their relation with torsion, is given in section 5. Finally, section 6 contains some conclusions. In addition, we report some conventions and several details of our computation in four appendices A–D.

2. Anomalies in heterotic string theory

In string theory, anomalous amplitudes happen to be total modular derivatives, and therefore receives contributions only from the boundaries of the moduli space of the relevant world-sheet surface. In heterotic models, for instance, anomalies are expressed as integrals over the boundary $\partial\mathcal{F}$ of the fundamental domain $\mathcal{F}$ of toroidal world-sheets, and their cancellation is then a direct consequence of modular invariance. This has been shown in detail in [29]–[32] for $D = 10$ heterotic theories.

The arguments of [29]–[32] can easily be generalized to $D = 4$ vacua. In order to do so, it is convenient to regard the anomaly as a possible non-vanishing amplitude involving an unphysical longitudinally polarized particle. The relevant string world-sheet is a torus. Moreover, since standard chiral anomalies only arise in the CP-odd part of the effective action, whereas others, such as target-space anomalies, distribute in a supersymmetric way between CP-even and CP-odd parts, it will be sufficient to restrict to correlations in the odd spin structure. Recall that in this

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completely periodic spin structure there is a world-sheet gravitino zero-mode inducing the insertion of a world-sheet supercurrent, the picture-changing operator T_F. Moreover, owing to the presence of a Killing spinor, one of the vertex operators must be taken in the so-called (-1)-picture, while all the others can be taken in the 0-picture [35]. It is easy to verify that anomalous amplitudes can arise only when the vertex operator associated to the longitudinal particle is the one in the (-1)-picture. However, the unphysical (-1)-picture vertex $V^{\mathrm{unphy}\cdot}$ is BRS-trivial, and can be rewritten as $Q\cdot\hat{V}^{\mathrm{unphy}\cdot}$ for some appropriate $\hat{V}^{\mathrm{unphy}\cdot}$. One can then bring Q to act on the rest of the correlation. Since the action of Q on the physical 0-picture vertices $V^{\mathrm{phy}\cdot}$ is trivial, its only effect is to convert T_F into the right-moving world-sheet energy-momentum tensor T_B: $Q\cdot T_F = T_B$. Finally, the net effect of the insertion of T_B is to produce the derivative with respect to the torus modulus $\bar{\tau}$ of the remaining correlation.[2] More precisely, one finds as expected a total derivative in moduli space. A generic anomalous amplitude is therefore of the form $\mathcal{A} = \oint_{\partial\mathcal{F}} d\tau \langle \hat{V}_1^{\mathrm{unphy}\cdot} V_2^{\mathrm{phy}\cdot}\cdots V_n^{\mathrm{phy}\cdot}\rangle$, the relevant number of vertex operators being determined by the integration over the Grassmann fermionic zero modes. Importantly, such an anomaly $\mathcal{A}$ satisfies the Wess-Zumino (WZ) consistency condition, and is the WZ descent of $I_{\mathcal{A}} = \oint_{\partial\mathcal{F}} d\tau \langle V_1^{\mathrm{phy}\cdot} V_2^{\mathrm{phy}\cdot}\cdots V_n^{\mathrm{phy}\cdot}\rangle$: $\mathcal{A} = 2\pi i I_{\mathcal{A}}^{(1)}$.[3] The total anomaly polynomial I encoding all the anomalous amplitudes of this type is then given by

$$I = \oint_{\partial\mathcal{F}} d\tau\, \bar{A}(\tau)\,, \tag{2.1}$$

where $\bar{A}(\tau)$ is the generating functional of odd spin structure correlation functions, i.e. the partition function computed by deforming the free action with the vertex operators appearing in anomalous amplitudes. The result (2.1) is completely general and its WZ descent produces the full set of anomalies. The integration over fermionic zero modes in $\bar{A}(\tau)$ automatically selects the components of appropriate degree in any space-time dimensionality.

From a string theory point of view, the vanishing of the total anomaly is a consequence of modular invariance [29]–[32]. Following [29], one can argue that the contribution from the part of $\partial\mathcal{F}$ closing at infinity vanishes by analytic continuation from kinematical regions where the correlation of the factors $\exp ip\cdot X$ in the vertex operators give an exponential suppression. The rest of the integral over $\partial\mathcal{F}$ vanishes instead under the condition that the generating functional $\bar{A}$ is modular-invariant:

$$\bar{A}(\tau+1) = \bar{A}(\tau)\,, \tag{2.2}$$

$$\bar{A}(-1/\tau) = \tau^2 \bar{A}(\tau)\,. \tag{2.3}$$

[2] For convenience, we take $\tau \leftrightarrow \bar{\tau}$ with respect to the standard notation for the heterotic string.

[3] We use here the standard WZ descent notation. Given a gauge-invariant $D+2$ form I, one can define a $D+1$ Chern-Simons form $I^{(0)}$ such that $I = dI^{(0)}$, whose variation is exact and defines a D-form $I^{(1)}$ through $\delta I^{(0)} = dI^{(1)}$.

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This can be easily understood from figure 1. Indeed, the contributions from parts I and IV of $\partial\mathcal{F}$ cancel thanks to (2.2), and similarly the contributions II and III cancel by virtue of (2.3). Finally, one finds therefore:

$$I = 0\,. \tag{2.4}$$

At the level of low-energy effective action, it is well-known that (2.4) appears as a cancellation between a one-loop anomaly and a tree-level Green-Schwarz inflow.

These two contributions can be explicitly identified in the low-energy limit $\alpha' \to 0$ of the above string theory computation. In fact, it was shown in [29] that the low-energy limit of $\bar{A}(\tau)$ is technically equivalent to its $\tau \to i\infty$ limit. Using this result, it is possible to disentangle the contribution corresponding to the one-loop anomaly from the contribution of the GS inflow, which is seen to factorize on a pole corresponding to the exchange of the NSNS two-form B.

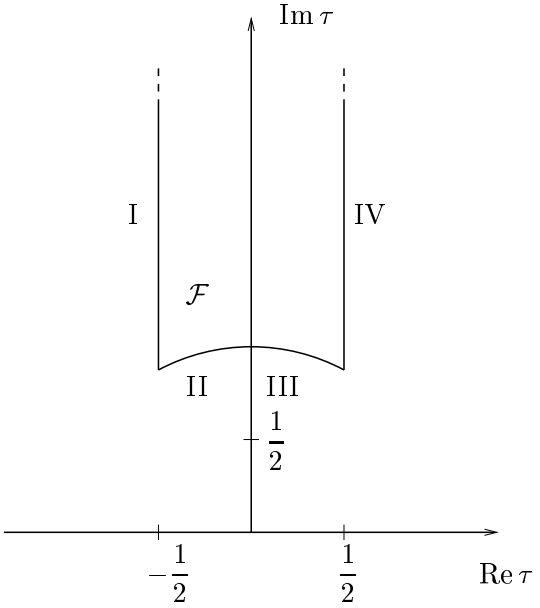

Figure 1: Fundamental domain of the torus.

The low-energy interpretation can be made more concrete by explicitly evaluating the generating functional $\bar{A}$ at leading order in $\alpha' \to 0$. In this limit, the path integral representation can be evaluated exactly, using a modular-invariant regularization. It becomes clear that $\bar{A}$ is intimately related to a character-valued index A, obtained from the same path-integral but with a holomorphic regularization. This is a generalization of the situation described in [26]–[28] to the case of orbifold theories with an additional target-space background. With a slight abuse of language, we will simply refer to A and $\bar{A}$ as elliptic genera in the following. On general grounds, the modular-invariant elliptic genus $\bar{A}$ differs from the holomorphic elliptic genus A only by a so-called modular anomaly involving some four-form X_4 [27]. The precise relation is

$$\bar{A}(\tau) = \exp\left\{-\frac{X_4}{64\pi^3\,\mathrm{Im}\tau}\right\} A(\tau)\,, \tag{2.5}$$

where X_4 is in fact entirely determined by the requirement of modular invariance, and can be computed in a direct way using a specific modular-invariant regularization prescription (see appendix D).

The appearance of the modular-invariant version $\bar{A}$ of the elliptic genus A in the expression for the anomaly polynomial is of course not a coincidence. Indeed, it is well known that its holomorphic companion A represents the chiral index of the full

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string spectrum, and the one-loop field-theory anomaly associated to chiral massless states is therefore given by [26]

$$I_{FT} = \lim_{\tau \to i\infty} A(\tau) = X_6 \ , \tag{2.6}$$

where the second equality anticipates that only the six-form component of the result is relevant in $D = 4$. The GS term, in turn, can be obtained by generalizing the work of [27] to the case at hand. Its expression involves the modular-invariant elliptic genus $\bar{A}$ and is given by

$$L_{GS} = B \wedge \left[\frac{1}{64\pi^2} \int_{\mathcal{F}} \frac{d^2\tau}{(\mathrm{Im}\tau)^2} \bar{A}(\tau) \right] , \tag{2.7}$$

where B is the NSNS two-form, dual to the axion in four dimensions. Using then the general expression (2.5) into (2.7), and going through the same manipulations as in [27], the modular integral yielding the GS term can easily be evaluated. In $D = 4$, one finds

$$L_{GS} = -2\pi\, B \wedge X_2 \ , \tag{2.8}$$

where the two-form X_2 is formally defined as $X_2 = X_6/X_4$. Clearly, this makes sense only if $X_6 = X_2 \wedge X_4$, i.e. only if the field theory anomaly factorizes. This must be guaranteed by the form (2.5) of the elliptic genus, that is modular invariance. The induced inflow of anomaly is $I_{GS} = -X_6$ if and only if the Bianchi identity satisfied by the field strength H of the B field is

$$dH = X_4 \, . \tag{2.9}$$

Summarizing, the vanishing of the total string theory anomaly is interpreted at low energies as the cancellation of the field-theory anomaly through the GS mechanism: $I_{GS} = -I_{FT}$. The generalization of the arguments of [30, 31, 32, 29] demonstrates that anomaly cancellation at the string-theory level is a consequence of modular invariance, whereas the generalization of the arguments of [27, 29] allows us to interpret the field-theory counterpart of the cancellation mechanism. Intuitively, one can think of the field-theory anomaly as the A part of $\bar{A}$, and attribute the GS inflow to the anomalous phase through which $\bar{A}$ differs from A.

As a remark, we would like to propose a very naive but suggestive alternative way to understand how the vanishing of the anomaly is achieved at the string-theory level. It is motivated by the unpleasant feature that the above arguments involve killing the contribution from infinity in $\partial\mathcal{F}$ by analytic continuation, whereas (2.6) suggests that this contribution should instead be directly linked to the field-theory anomaly. The point is that the field-theory Schwinger parameter associated to the modulus τ is actually $\alpha'\tau$, so that the various contributions to the anomaly appear from potentially different corners of moduli space, depending on whether one takes $\alpha' \to 0$ from the beginning or only at the end. This is also the essence of the

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already mentioned observation of [29] about the technical equivalence of the two limits $\alpha' \to 0$ and $\tau \to i\infty$ in anomalous amplitudes. In particular, if one takes brutally $\alpha' \to 0$ from the beginning, one can no longer kill the contribution at infinity in $\partial\mathcal{F}$ by analytic continuation, since each pair of momenta is accompanied by an α'. Similarly, the term responsible for the factorization of the amplitude (2.1) on a B-pole is suppressed. These two effects seem to compensate each other, and as a matter of fact, one could recover a natural interpretation of the cancellation mechanism by taking the $\alpha' \to 0$ from the beginning and computing the elliptic genus with a holomorphic rather than modular-invariant regularization. The total anomaly (2.1) would then involve A instead of $\bar{A}$, and the vanishing of the contour integral would be attributed to holomorphicity of A in $\mathcal{F}$, rather than modular invariance. However, the relevant pieces of $\partial\mathcal{F}$ are now different. The contribution at infinity yields the field-theory anomaly. The contributions from I and IV still cancel, because (2.2) is still satisfied for A. The contributions from II and III, instead, no longer cancel since (2.3) fails for A, but rather combine to yield the GS inflow. Although this alternative argument cannot be taken too seriously, we find it quite suggestive.

3. Orbifolds and elliptic indices

In this section, we shall provide a concrete example of the general features discussed in the previous section, by studying all possible gauge/gravitational/target-space anomalies for $\mathbb{Z}_N$ orbifold models with standard embedding and N odd, i.e. the $\mathbb{Z}_3$ and $\mathbb{Z}_7$ models. According to the arguments above, this involves computing the elliptic genus in a gauge, gravitational and target-space background, in a sense that we shall now make more precise.

To begin with, let us briefly recall some basic facts about $\mathbb{Z}_N$ heterotic orbifolds. The generator of the orbifold action is defined by a twist vector v_i, where $i = 1, 2, 3$ label the three internal tori and the associated complex coordinates, satisfying the condition $\sum_i v_i = 0$. For standard embedding, the orbifold action on the gauge lattice is simply a shift by v_i itself and acts on an SU(3) part of the Cartan subgroup SO(16) of one of the two E_8 factors ($v_j = 0$ for $j > 3$). The total gauge group is then $E_8 \times E_6 \times H$, where H is a subgroup of SU(3) (see table 1).

It is well known that these heterotic vacua possess a target-space duality symmetry. This symmetry acts as $\mathrm{SL}(2,\mathbb{R})_i$ transformations on the three universal diagonal Kähler moduli superfields T_i [14]:

$$T_i \to \frac{a_i T_i - i b_i}{i c_i T_i + d_i} \,. \tag{3.1}$$

Its discrete $\mathrm{SL}(2,\mathbb{Z})$ subgroup is an exact symmetry (to all orders in α' and g_s) of the model, T-duality. At tree level, (3.1) is a symmetry of the low-energy $N = 1$

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supergravity effective action, provided one transforms in an appropriate way each matter chiral superfield as well, requiring:

$$\Phi_s \to \exp\Big\{ - n_i \ln(ic_i T_i + d_i)\Big\}\Phi_s \ , \tag{3.2}$$

with given coefficients n_i, the so-called modular weights [20]. Since the transformations (3.1) and (3.2) act as chiral rotations on the component fermions, this symmetry is potentially anomalous. At one loop, anomalies appear through triangular graphs where the connection Z_i associated to the target-space symmetry (3.1) appears, possibly together with gluons and gravitons, among the external states (see [18]–[21]). Although such anomalies (mainly mixed ones) have received quite a lot of attention in the past, no direct, explicit and complete computation of them has so far appeared in the literature. A satisfactory understanding of their structure is also missing. Since the target-space connections Z_i are composite and not elementary fields (see (3.16)), such anomalies are typically derived by analogy with standard gauge anomalies associated to an elementary connection. This approach was adopted also in [21], the only reference considering target-space/gauge/gravitational anomalies in full generality.

The approach we follow here is more direct: we will compute anomalous amplitudes involving the elementary constituents of the composite connections Z_i. We focus on the dependence on the diagonal untwisted moduli. These are defined as:

$$T_i = G_{i\bar{\imath}} + iB_{i\bar{\imath}} \, , \tag{3.3}$$

where the scalars $G_{i\bar{\imath}}$ and the pseudoscalars $B_{i\bar{\imath}}$ represent the internal metric and B-field components of the orbifold in complex coordinates along each of the three T_i^2 tori. It is straightforward to determine the most general form of the target-space connections Z_i and their associated field strengths G_i in terms of these fields. The total connection one-form is $Z = \sum_i (\alpha_i dT_i + \bar{\alpha}_{\bar{\imath}} d\overline{T}_{\bar{\imath}})$, where the α_i's are some functions of the T-moduli. Correspondingly, the general form of the field strength $G = dZ$ is

$$G = \frac{1}{2}\sum_{i,j}\Big[\alpha_{i,j}\, dT_i \wedge dT_j + \bar{\alpha}_{\bar{\imath},\bar{\jmath}}\, d\overline{T}_{\bar{\imath}} \wedge d\overline{T}_{\bar{\jmath}} + (\alpha_{i,\bar{\jmath}} - \bar{\alpha}_{\bar{\jmath},i})\, dT_i \wedge d\overline{T}_{\bar{\jmath}}\Big] \, . \tag{3.4}$$

From (3.4), it is clear that the composite field strength is always at least quadratic in the fluctuations of the moduli (3.3). Correspondingly, one can reliably compute anomalous triangular graphs with external composite fields by replacing each of them with a couple of T fields. Actually, it is known from supergravity that the coefficients α_i are determined by the Kähler potential K as $\alpha_i = -i\partial K/\partial T_i = -iK_i$.[4] The field

[4]The covariant derivative of a generic chiral multiplet contains the sum of the Kähler and the σ-model connections, proportional to each other through the modular weight. The target-space connection is identified with the Kähler connection, and the total charge of each state is $1 + 2n_i$.

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strength (3.4) should therefore reduce to:

$$G = -i \sum_{i,j} K_{i,\bar{\jmath}}\, dT_i \wedge d\overline{T}_{\bar{\jmath}}\,. \tag{3.5}$$

Finally, the relevant part of the Kähler potential for the models under consideration is $K = -\sum_i \ln(T_i + \overline{T}_{\bar{\imath}})$, yielding a diagonal result: $K_{i,\bar{\jmath}} = \delta_{i,\bar{\jmath}}/2(T_i + \overline{T}_{\bar{\imath}})^2$. In the following, we shall assume only the general form (3.4), and verify that the supergravity result (3.5) is correctly recovered.

In order to evaluate the elliptic genus as a function of the gauge, gravitational and target-space backgrounds, we need to analyse the corresponding gluon, graviton and moduli vertex operators. According to the general discussion of section 2, we have to take into account only physical operators in the 0-picture. Consider first the standard case of photons and gravitons. Denoting with J^a the current operator of the gauge current algebra, and with Q_a its zero mode, one has:

$$\begin{aligned} V_g &= \xi_{\mu\nu}(p) \int d^2z\, \partial X^\mu(z)\, \left(\bar\partial X^\nu + ip\cdot\psi\psi^\mu\right)(\bar z)\, e^{ip\cdot X(z,\bar z)}\,, \\ V_\gamma &= \xi^a_\mu(p) \int d^2z\, J^a(z)\, \left(\bar\partial X^\mu + ip\cdot\psi\psi^\mu\right)(\bar z)\, e^{ip\cdot X(z,\bar z)}\,. \end{aligned} \tag{3.6}$$

In the low-energy limit $\alpha' \to 0$, one can restrict to the terms providing a minimal number of momenta and a maximal number of fermionic zero-modes, and one finds:

$$\begin{aligned} V_g^{\text{eff.}}(R) &= R_{\mu\nu} \int d^2z\, X^\mu(z)\, \partial X^\nu(z)\,, \\ V_\gamma^{\text{eff.}}(F) &= F^a\, Q^a\,, \end{aligned} \tag{3.7}$$

where F^a and $R_{\mu\nu}$ represent the gauge and gravitational curvatures and are given by

$$F^a = \frac{1}{2}\, F^a_{\mu\nu}\, \psi_0^\mu \psi_0^\nu\,, \qquad R_{\mu\nu} = \frac{1}{2}\, R_{\mu\nu\rho\sigma}\, \psi_0^\rho \psi_0^\sigma\,. \tag{3.8}$$

Consider next the case of the T_i moduli. Their vertex operators are given by:[5]

$$V_{T_i} = T_i(p) \int d^2z\, \partial\overline{X}^{\bar\imath}(z) \Big(\bar\partial X^i + ip\cdot\psi\psi^i\Big)(\bar z)\, e^{ip\cdot X(z,\bar z)}\,, \tag{3.9}$$

$$V_{\overline{T}_{\bar\imath}} = \overline{T}_{\bar\imath}(\bar p) \int d^2z\, \partial X^i(z) \Big(\bar\partial\overline{X}^{\bar\imath} + i\bar p\cdot\psi\overline{\psi}^{\bar\imath}\Big)(\bar z)\, e^{i\bar p\cdot X(z,\bar z)}\,, \tag{3.10}$$

and lead to:

$$V_{T_i}^{\text{eff.}} = dT_i \int d^2z\, \psi^i(\bar z)\, \partial\overline{X}^{\bar\imath}(z)\,, \tag{3.11}$$

$$V_{\overline{T}_{\bar\imath}}^{\text{eff.}} = d\overline{T}_{\bar\imath} \int d^2z\, \overline{\psi}^{\bar\imath}(\bar z)\, \partial X^i(z)\,, \tag{3.12}$$

[5] In (3.10), $\bar p$ denotes simply the momentum of $\bar T$, not the complex conjugate of p.

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where

$$dT_i = ip_\mu T_i\,\psi_0^\mu\,, \qquad d\overline{T}_{\bar\imath} = i\bar p_\mu \overline{T}_{\bar\imath}\,\psi_0^\mu\,. \tag{3.13}$$

It is clear from (3.11) and (3.12) that the only non-vanishing contractions among moduli occur between a given T_i modulus and its complex conjugate $\overline{T}_{\bar\imath}$. A generic correlation is therefore non-vanishing only if it includes an equal number of T_i and $\overline{T}_{\bar\imath}$ vertices. This general property can be easily understood also in the path-integral representation of the generating functional. Indeed, integrating out the internal fermions appearing in (3.11) and rescaling the internal bosonic fields to normalize their kinetic terms, one ends up with an effective bosonic interaction that is interpreted as the effective vertex operator for the composite connection. This manipulation is equivalent to grouping in all possible ways the moduli vertices in T_i-$\overline{T}_{\bar\imath}$ pairs and explicitly performing the fermionic contraction within each pair. One finds:

$$V_{Z_i}^{\rm eff.}(G_i) = G_i \int d^2z\,\overline{X}^{\bar\imath}(z)\,\partial X^i(z)\,, \tag{3.14}$$

with G_i representing the target-space curvature two-forms and given by

$$G_i = \frac{i}{2(T_i+\overline{T}_{\bar\imath})^2}\,p_\mu\,T_i\,\bar p_\nu\,\overline{T}_{\bar\imath}\,\psi_0^\mu\,\psi_0^\nu\,. \tag{3.15}$$

As expected, the external moduli automatically group in T_i-$\overline{T}_{\bar\imath}$ pairs, reconstructing composite connections of the form (3.5) with the correct Kähler potential. Correpondingly, one can define three independent connections and curvatures associated to each internal torus and given by

$$Z_i = \frac{-i}{(T_i+\overline{T}_{\bar\imath})}\,d(T_i-\overline{T}_{\bar\imath})\,, \tag{3.16}$$

$$G_i = \frac{-i}{2(T_i+\overline{T}_{\bar\imath})^2}\,dT_i\wedge d\overline{T}_{\bar\imath}\,. \tag{3.17}$$

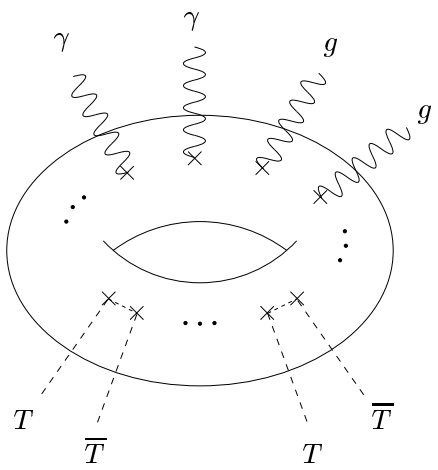

Figure 2: Pairing of moduli in anomalous amplitudes.

It is important to stress that within this direct approach, the pairing of the moduli appearing as external states into composite connections is a derived property and not an assumption. This mechanism is depicted in figure 2 for a generic anomalous amplitude.

The computation of the elliptic genus does not present any problem, since all the effective vertices are quadratic. We begin with the holomorphic regularization. The result factorizes into space-time, compact and gauge partition functions:

$$A(\tau;F,R,G) = \frac{1}{2N}\sum_{k,l=0}^{N-1} N_{k,l}\,Z_{ST}(R)\,Z_C^{k,l}(G)\,Z_\Gamma^{k,l}(F)\,. \tag{3.18}$$

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The $1/2N$ factor is due to the GSO projection in a $\mathbb{Z}_N$ orbifold model and $N_{k,l}$ represents the number of points that are at the same time k- and l-fixed. Restricting to the six-form component of the elliptic genus relevant in four dimensions, one can rescale the free-particle normalization to cancel the τ-dependence coming from the curvatures. The only τ dependence that is left over arises then through $q = e^{2i\pi\tau}$. For $(k,l) \neq (0,0)$, one finds the following results in terms of the skew eigenvalues y_i, λ_a and g_i of the rescaled gauge, gravitational and target space curvatures (see appendix A) (recall that $v_j = 0$ for $j > 3$):[6]

$$Z_{ST}(R) = \eta^2(\tau) \prod_{a=1}^{2} \frac{(i\lambda_a)\eta(\tau)}{\theta_1(i\lambda_a/2\pi|\tau)}\,, \tag{3.19}$$

$$Z_C^{k,l}(G) = \prod_{i=1}^{3} \frac{\eta(\tau)}{\theta_1\left[\begin{smallmatrix} lv_i \\ kv_i \end{smallmatrix}\right](-g_i/2\pi|\tau)}\,, \tag{3.20}$$

$$Z_\Gamma^{k,l}(F) = \frac{1}{4} \sum_{\alpha,\beta} \prod_{i=1}^{8} \frac{\theta_\alpha\left[\begin{smallmatrix} lv_i \\ kv_i \end{smallmatrix}\right](-y_i/2\pi|\tau)}{\eta(\tau)} \prod_{h=1}^{8} \frac{\theta_\beta(-y'_h/2\pi|\tau)}{\eta(\tau)}\,. \tag{3.21}$$

In the completely untwisted sector $(k,l) = (0,0)$, there are six additional fermionic zero modes arising from the compact dimensions, and the result is apparently vanishing. Actually, there is a potential contribution from this sector to pure target-space anomalies of the form $G_1G_2G_3$. This can be easily understood by noting from (3.11) and (3.12) that in this case it is possible to soak the internal fermionic zero modes as well. However, the correlation that one obtains involves only internal right-moving momenta, and is clearly irrelevant in the limit (2.6) and for the exponential term in (2.5). Hence, it will be completely neglected in the following.[7]

The corresponding result in the modular-invariant regularization summarized in appendix D is found to have the expected form (2.5) with

$$X_4(F,R,G) = 8\pi^2 \left(\sum_{a=1}^{2} \lambda_a^2 + \sum_{i=1}^{3} g_i^2 - \sum_{i=1}^{8} y_i^2 - \sum_{h=1}^{8} {y'_h}^2 \right). \tag{3.22}$$

This is just what is needed for $\bar{A}$ to satisfy the modular transformation properties (2.2) and (2.3). Indeed, it is not difficult, although a little bit laborious, to verify that:

$$\bar{A}(\tau+1;F,R,G) = \bar{A}(\tau;F,R,G)\,, \tag{3.23}$$

$$\bar{A}\left(-\frac{1}{\tau};F,R,G\right) = \frac{1}{\tau}\bar{A}(\tau;F\tau,R\tau,G\tau) = \tau^2\bar{A}(\tau;F,R,G)\,, \tag{3.24}$$

where the last step in the second equation holds for the relevant six-form component.

[6] The definition of twisted θ-functions used here is taken from [7, appendix A].

[7] From a more physical point of view, it is clear that the completely untwisted $(0,0)$ sector, having $N = 4$ supersymmetry, cannot give rise to any anomaly.

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Before entering into the details of the expansion of eqs. (3.19)–(3.21), it is convenient to define the quantity

$$C_k = \prod_{i=1}^{3} \sin \pi k v_i \,, \tag{3.25}$$

and the representative

$$\theta_i^l = l v_i - \mathrm{int}(l v_i) \tag{3.26}$$

of the generic twist vector $l v_i$ in the interval $[0,1]$. Whereas $\sum_i v_i$ is 0 in order to have $N = 1$ supersymmetry, $\sum_i \theta_i^l$ is 1 for half of the twisted sectors and 2 for the conjugate half. More precisely, one can define the sign ϵ_l distinguishing between conjugate twisted sectors as

$$\epsilon_l = -(-1)^{\sum_i \theta_i^l} = -\mathrm{sign}(C_l) \,. \tag{3.27}$$

We also define S_l to be the set of twisted sectors where $\epsilon_l = 1$. The set S_l is $\{1\}$ for $\mathbb{Z}_3$ and $\{1,2,4\}$ for $\mathbb{Z}_7$.

3.1 Field theory anomaly

The field theory anomaly I_{FT} can now be computed according to (2.6). Taking the limit $\tau \to i\,\infty$ of A turns out to be a rather non-trivial exercise. For this reason, we present here only the final result, leaving the relevant details to appendices B and C. As expected, simple characteristic classes, defined in appendix A, are reconstructed. Since the visible gauge group is $E_6 \times H$, with $H \subset \mathrm{SU}(3)$ depending on the orbifold projection, it is useful to refer to the underlying $E_6 \times \mathrm{SU}(3)$ group common to all orbifold models with standard embedding. In the untwisted sector, one finds

$$\begin{aligned} I_{FT}^{\mathrm{untw.}}(F,R,G) = \frac{1}{2N} \sum_{k=1}^{N-1} C_k \Big[& \widehat{A}_k(G)\, \widehat{G}(R) + \widehat{G}_k(G)\, \widehat{A}(R) + \\ & + \Big(\mathrm{ch}_{\mathbf{248}}(F) + \mathrm{ch}_{(\mathbf{78},\mathbf{1})}(F) + \mathrm{ch}^k_{(\mathbf{1},\mathbf{8})}(F) + \\ & + \mathrm{ch}^k_{(\mathbf{27},\mathbf{3})}(F) + \mathrm{ch}^k_{(\overline{\mathbf{27}},\overline{\mathbf{3}})}(F) \Big) \widehat{A}_k(G)\, \widehat{A}(R) \Big] \,, \end{aligned} \tag{3.28}$$

in terms of the twisted Chern characters $\mathrm{ch}^k_\rho(F)$ defined in (A.5), (A.6). In the twisted sectors, finds instead the following compact expression for the anomaly:

$$\begin{aligned} I_{FT}^{\mathrm{tw.}}(F,R,G) = -i \sum_{l \in S_l} N_l \Big[& \mathrm{ch}_{(\mathbf{27},\mathbf{1})}(F)\ \mathrm{ch}_{\theta_i^l - \frac{1}{3}}(F)\ \mathrm{ch}_{-(1-\theta_i^l)}(G) + \\ & + \sum_s \mathrm{ch}_{q_i(s)}(F)\ \mathrm{ch}_{n_i(s)}(G) \Big] \widehat{A}(R) \,, \end{aligned} \tag{3.29}$$

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P	v	n_T	H	Matter	
$\mathbb{Z}_3$	$(\frac{1}{3},\frac{1}{3},\frac{-2}{3})$	9	SU(3)	$1:(\mathbf{27},\mathbf{3})_{-\delta_i^{1,2,3}}$	$g:(\mathbf{27},\mathbf{1})_{-\rho_i^{1,2,3}}$ $(\mathbf{1},\overline{\mathbf{3}})_{-(\rho_i^1+\delta_i^{1,2,3})}$
$\mathbb{Z}_7$	$(\frac{1}{7},\frac{2}{7},\frac{-3}{7})$	3	$\mathrm{U}(1)^2$	$1:(\mathbf{27}_{0,2})_{-\delta_i^1}$ $(\mathbf{27}_{1,-1})_{-\delta_i^2}$ $(\mathbf{27}_{1,3})_{-\delta_i^3}$ $(\mathbf{1}_{1,-3})_{-\delta_i^1}$ $(\mathbf{1}_{-2,0})_{-\delta_i^2}$ $(\mathbf{1}_{-1,-1})_{-\delta_i^3}$	$g:(\mathbf{27}_{\frac{-2}{7},\frac{-4}{7}})_{-\rho_i^1}$ $(\mathbf{1}_{\frac{-2}{7},\frac{-18}{7}})_{-(\rho_i^1+\delta_i^1)}$ $(\mathbf{1}_{\frac{-9}{7},\frac{3}{7}})_{-(\rho_i^1+\delta_i^2)}$ $(\mathbf{1}_{\frac{5}{7},\frac{3}{7}})_{-(\rho_i^1+\delta_i^3)}$ $(\mathbf{1}_{\frac{-9}{7},\frac{3}{7}})_{-(\rho_i^1+2\delta_i^1)}$ $(\mathbf{1}_{\frac{5}{7},\frac{3}{7}})_{-(\rho_i^1+2\delta_i^2)}$ $(\mathbf{1}_{\frac{5}{7},\frac{3}{7}})_{-(\rho_i^1+\delta_i^1-\delta_i^3)}$ $(\mathbf{1}_{\frac{5}{7},\frac{3}{7}})_{-(\rho_i^1+2\delta_i^1+\delta_i^2)}$ $(\mathbf{1}_{\frac{5}{7},\frac{3}{7}})_{-(\rho_i^1+4\delta_i^1)}$
				$g^2:(\mathbf{27}_{\frac{3}{7},\frac{-1}{7}})_{-\rho_i^2}$ $(\mathbf{1}_{\frac{10}{7},\frac{6}{7}})_{-(\rho_i^2+\delta_i^3)}$ $(\mathbf{1}_{\frac{3}{7},\frac{-15}{7}})_{-(\rho_i^2+\delta_i^1)}$ $(\mathbf{1}_{\frac{-4}{7},\frac{6}{7}})_{-(\rho_i^2+\delta_i^2)}$ $(\mathbf{1}_{\frac{3}{7},\frac{-15}{7}})_{-(\rho_i^2+2\delta_i^3)}$ $(\mathbf{1}_{\frac{-4}{7},\frac{6}{7}})_{-(\rho_i^2+2\delta_i^1)}$ $(\mathbf{1}_{\frac{-4}{7},\frac{6}{7}})_{-(\rho_i^2+\delta_i^3-\delta_i^2)}$ $(\mathbf{1}_{\frac{-4}{7},\frac{6}{7}})_{-(\rho_i^2+2\delta_i^3+\delta_i^1)}$ $(\mathbf{1}_{\frac{-4}{7},\frac{6}{7}})_{-(\rho_i^2+4\delta_i^3)}$	$g^4:(\mathbf{27}_{\frac{-1}{7},\frac{5}{7}})_{-\rho_i^4}$ $(\mathbf{1}_{\frac{-8}{7},\frac{12}{7}})_{-(\rho_i^4+\delta_i^2)}$ $(\mathbf{1}_{\frac{6}{7},\frac{12}{7}})_{-(\rho_i^4+\delta_i^3)}$ $(\mathbf{1}_{\frac{-1}{7},\frac{-9}{7}})_{-(\rho_i^4+\delta_i^1)}$ $(\mathbf{1}_{\frac{6}{7},\frac{12}{7}})_{-(\rho_i^4+2\delta_i^2)}$ $(\mathbf{1}_{\frac{-1}{7},\frac{-9}{7}})_{-(\rho_i^4+2\delta_i^3)}$ $(\mathbf{1}_{\frac{-1}{7},\frac{-9}{7}})_{-(\rho_i^4+\delta_i^2-\delta_1^i)}$ $(\mathbf{1}_{\frac{-1}{7},\frac{-9}{7}})_{-(\rho_i^4+2\delta_i^2+\delta_3^i)}$ $(\mathbf{1}_{\frac{-1}{7},\frac{-9}{7}})_{-(\rho_i^4+4\delta_i^2)}$

Table 1: The $\mathbb{Z}_3$ and $\mathbb{Z}_7$ models, with their twist v, enhancement gauge group H, number n_T of Kähler moduli and matter spectrum. The representations refer to the visible gauge group $G = E_6 \times H$ and have multiplicity equal to the number of fixed points in each twisted sector. The modular weights n_i of each matter field appear as a subindex ($\rho_i^l = 1 - \theta_i^l$).

where the last sum runs over all the massless E_6-neutral twisted states s, with modular weights $n_i(s)$ and SU(3) charge vector $q_i(s)$. The structure of these states is described in more detail in appendix C. Their contributions can be read off directly from the elliptic genus, and the correct modular weights and charge vectors are found. The Chern characters $\mathrm{ch}_{q_i}(F)$ and $\mathrm{ch}_{n_i}(G)$ are defined in (A.6) and (A.7), and the complete massless spectra are reported in table 1.

As discussed in last section, consistency requires that the total anomaly must take a factorized form, as a consequence of modular invariance. We will now verifiy this explicitly for the two models under consideration. Irreducible gauge-anomalies cancel between untwisted an twisted sectors. Similarly, the structure of the cubic target-space anomalies drastically simplifies in the total results, in such a way to allow a simple factorization compatible with modular invariance.

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$\mathbb{Z}_3$ model. For the $\mathbb{Z}_3$ model, $H = \mathrm{SU}(3)$. Consider first the untwisted sector. The twisted Chern characters appearing in (3.28) are easily written in terms of the ordinary ones. Defining $\alpha = \exp 2\pi i/3$, one finds:[8]

$$I_{FT}^{\text{untw.}} = \frac{1}{6}\sum_{k=1}^{2} C_k \Big[\widehat{A}_k(G)\,\widehat{G}(R) + \widehat{G}_k(G)\,\widehat{A}(R) + \\ + \Big(\mathrm{ch}_{\mathbf{248}}(F) + \mathrm{ch}_{(\mathbf{78},\mathbf{1})}(F) + \mathrm{ch}_{(\mathbf{1},\mathbf{8})}(F) + \\ + 2\,\alpha^k\,\mathrm{ch}_{(\mathbf{27},\mathbf{3})}(F) \Big)\,\widehat{A}_k(G)\,\widehat{A}(R) \Big] . \tag{3.30}$$

Consider next the twisted sector. As can be seen from table 1, there is a triplet of non-oscillator states, and from (3.29) one gets:

$$I_{FT}^{\text{tw;}}(F,R,G) = -27\,i \left[\mathrm{ch}_{(\mathbf{27},\mathbf{1})}(F)\ \mathrm{ch}_{-(1-\theta_i^1)}(G) + \sum_{j=1}^{3} \mathrm{ch}_{(\mathbf{1},\overline{\mathbf{3}})}(F)\,\mathrm{ch}_{-(1-\theta_i^1+\delta_i^j)}(G) \right] \widehat{A}(R)\,. \tag{3.31}$$

It is then straightforward to compute the explicit expression of the total anomaly. One finds as expected a factorized expression:

$$I_{FT} = \frac{15}{2(2\pi)^3} \left(\sum_{i=1}^{3} G_i \right) \left(\mathrm{tr}\, R^2 - \mathrm{tr}_{E_8}\, F^2 - \frac{1}{3}\,\mathrm{tr}_{E_6}\, F^2 - 2\,\mathrm{tr}_{\mathrm{SU}(3)}\, F^2 + 2\sum_{i=1}^{3} G_i^2 \right) . \tag{3.32}$$

We have used the standard definition $\mathrm{Tr}_{E_8} F^2 = 30\,\mathrm{tr}_{E_8}\, F^2$ for the **248** of E_8 and the relation $\mathrm{Tr}_{E_6} F^2 = 4\,\mathrm{tr}_{E_6}\, F^2$ between the **78** and **27** of E_6.

$\mathbb{Z}_7$ model. For the $\mathbb{Z}_7$ model, $H = \mathrm{U}(1) \times \mathrm{U}(1)$. The definition of the embedding of $\mathrm{U}(1) \times \mathrm{U}(1) \subset \mathrm{SU}(3)$ is completely arbitrary, and for convenience we will not yet make any precise choice. In fact, together with the $\mathrm{U}(1) \subset E_6$, there are three $\mathrm{U}(1)_i$, $i = 1, 2, 3$, representing the unbroken part of the SO(6) group of internal rotations. There is then a natural choice for this subgroup, in which $\mathrm{U}(1)_i$ corresponds to rotations within the i-th internal torus T_i^2. The i-th component of the charge vector q_i, introduced above for a generic state contributing to the anomaly, then represents the charge with respect to this $\mathrm{U}(1)_i$. One can therefore keep the charge vector q_i to describe the H quantum numbers; since q_i satisfies $\sum_i q_i = 0$, it is always in SU(3) and never provides additional charge under $\mathrm{U}(1) \subset E_6$. We will adopt this notation below, as was also done in [36].

Consider first the untwisted sector. Again, the twisted Chern characters are easily reduced to the ordinary ones. In particular, one finds three replicas of matter contributions, which it will prove convenient to label with $l = 1, 2, 4$ in the same

[8] We have that $\alpha^k\,\mathrm{ch}_R(F)$ and $\alpha^{-k}\,\mathrm{ch}_R(-F)$ give the same contribution when summing over k.

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way as the three twisted sectors, and define the corresponding permutation degree d_l such that $d_1 = 0$, $d_2 = 1$, $d_4 = 2$. Defining then $\alpha = \exp 2\pi i/7$, one finds:

$$I_{FT}^{\text{untw.}} = \frac{1}{14}\sum_{k=1}^{6} C_k \Bigg[\widehat{A}_k(G)\,\widehat{G}(R) + \widehat{G}_k(G)\,\widehat{A}(R) + \\ + \Bigg(\text{ch}_{\mathbf{248}}(F) + \text{ch}_{(\mathbf{78},\mathbf{1})}(F) + 2 + \\ + 2\sum_{l=1,2,4}\alpha^{lk}\left(\text{ch}_{\mathbf{27}_{-\delta_i^{1+d_l}+\delta_i^{2+d_l}}}(F) + \text{ch}_{\mathbf{1}_{\delta_i^{d_l}}}(F)\right)\Bigg)\widehat{A}_k(G)\,\widehat{A}(R)\Bigg]\,. \tag{3.33}$$

Consider next the twisted sectors. The three sectors $l = 1, 2, 4$ are related by permutations, and in each of them there are eight massless states, neutral under E_6, whose modular weights and charges (with a specific choice of U(1) charges, see below) are reported in table 1. The form of the anomaly then is

$$I_{FT}^{\text{tw.}}(F,R,G) = -7\,i\sum_{l=1,2,4}\Bigg[\text{ch}_{\mathbf{27}_{\theta_i-\frac{1}{3}}}(F)\ \text{ch}_{-(1-\theta_i)}(G) + \\ + \sum_{j=1,2,3}\text{ch}_{\mathbf{1}_{\theta_i-\delta_i^{j-d_l}}}(F)\ \text{ch}_{-(1-\theta_i+\delta_i^{j-d_l})}(G) + \\ + \sum_{h=1,2}\text{ch}_{\mathbf{1}_{\theta_i-\delta_i^{h-d_l+1}}}(F)\ \text{ch}_{-(1-\theta_i+2\delta_i^{h-d_l})}(G) + \\ + \text{ch}_{\mathbf{1}_{\theta_i-\delta_i^{3-d_l}}}(F)\ \text{ch}_{-(1-\theta_i+4\delta_i^{3-d_l})}(G) + \\ + \text{ch}_{\mathbf{1}_{\theta_i-\delta_i^{3-d_l}}}(F)\ \text{ch}_{-(1-\theta_i+2\delta_i^{1-d_l}+\delta_i^{2-d_l})}(G) + \\ + \text{ch}_{\mathbf{1}_{\theta_i-\delta_i^{3-d_l}}}(F)\ \text{ch}_{-(1-\theta_i+\delta_i^{1-d_l}-\delta_i^{3-d_l})}(G)\Bigg]\,\widehat{A}(R)\,. \tag{3.34}$$

In order to compute the explicit form of (3.34), one has to choose a specific embedding of the U(1)'s. One can define them through the corresponding charges $Q_{1,2}$ assigned to the generic SU(3) vector q_i. We take the combinations $Q_1(q) = (q_2 - q_3)/2\sqrt{2}$ and $Q_2(q) = (2q_1 - q_2 - q_3)/2\sqrt{6}$, which are canonically normalized. Notice that the (non-canonically normalized) charge Q_0 with respect to the $\text{U}(1) \subset E_6$ is given by $Q_0(q) = 2(q_1 + q_2 + q_3)$, which vanishes for any q_i occurring here. The charges obtained in this way for each state are reported in table 1, in units of $1/2\sqrt{2}$ for Q_1 and $1/2\sqrt{6}$ for Q_2. After a straightforward but lengthy computation, one finds:

$$I_{FT} = \frac{15}{2(2\pi)^3}\left(\sum_{i=1}^{3} G_i\right)\left(\text{tr}\,R^2 - \text{tr}_{E_8}\,F^2 - \frac{1}{3}\,\text{tr}_{E_6}\,F^2 - \frac{1}{2}\,F_1^2 - \frac{1}{2}\,F_2^2 + 2\sum_{i=1}^{3} G_i^2\right). \tag{3.35}$$

Notice that all U(1) anomalies cancel.

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3.2 GS terms

The GS term can be computed from (2.7), as described in last section. To do so, one has first of all to evaluate (3.22) in each model: the GS term is then unambiguously determined. One finds the following results for the two models under consideration:

$$\mathbb{Z}_3 \,:\; X_4 = \mathrm{tr}\, R^2 - \mathrm{tr}_{E_8}\, F^2 - \frac{1}{3}\,\mathrm{tr}_{E_6}\, F^2 - 2\,\mathrm{tr}_{\mathrm{SU}(3)}\, F^2 + 2\sum_{i=1}^{3} G_i^2\,, \tag{3.36}$$

$$\mathbb{Z}_7 \,:\; X_4 = \mathrm{tr}\, R^2 - \mathrm{tr}_{E_8}\, F^2 - \frac{1}{3}\,\mathrm{tr}_{E_6}\, F^2 - \frac{1}{2}\,F_1^2 - \frac{1}{2}\,F_2^2 + 2\sum_{i=1}^{3} G_i^2\,. \tag{3.37}$$

The gauge part of (3.36) and (3.37) is universal, i.e. independent of the gauge group factor, as required for a GS mechanism involving only the NSNS axion [19]. This can be easily verified by using the values $C(E_8) = 30$, $C(E_6) = 12$ and $C(\mathrm{SU}(3)) = 3$ for the quadratic Casimirs to evaluate gauge traces; one finds $\mathrm{tr}_{E_8}\, F^2 = 1/2\, F_{E_8}{}^n \wedge F_{E_8 n}$, $1/3\,\mathrm{tr}_{E_6}\, F^2 = 1/2\, F_{E_6}{}^n \wedge F_{E_6 n}$, $2\,\mathrm{tr}_{\mathrm{SU}(3)}\, F^2 = 1/2\, F_{\mathrm{SU}(3)}{}^n \wedge F_{\mathrm{SU}(3)_n}$, where the index n is summed over all the generators of the particular group factor.

Notice that the explicit form of X_4 in both models requires a very special factorization of the one-loop anomalies (3.32) and (3.35). For instance, cubic $G_1 G_2 G_3$ anomalies have to vanish, as indeed happens in both models, but thanks to a very non-trivial compensation of the contributions of all fields.

4. Low-energy interpretation

We shall now analyse the results of the string theory computation of last section from a low-energy supergravity point of view. In particular, we will attempt to make contact with the results of [19]–[21].

A first important issue is the form of the anomalies derived within string theory, eqs. (3.28) and (3.29). As we shall now discuss, the results that we obtain do not seem to match in all details the expectations based on a low-energy supergravity approach. A number of important qualitative differences can be understood and recognized to correspond to the different frames used in the two contexts: the Einstein and the string frames. The dependence on the gauge and gravitational curvatures is encoded in the standard characteristic classes, as expected. As for the dependence on the target-space curvature, on the other hand, there is an important qualitative difference between the contributions of untwisted and twisted states; for the former, this dependence resembles a gravitational dependence, whereas for the latter it is more similar to a gauge dependence. This suggest that twisted states feel the target-space background only through the target-space connection $Z_\mu = -i\sum_i (T_i + \overline{T}_{\bar{\imath}})^{-1}\partial_\mu (T_i - \overline{T}_{\bar{\imath}})$, whereas untwisted states have some additional sensitivity to it. In fact, this is recognized to be nothing but the effect of the moduli-dependent compactification volume

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$V = \prod_i (T_i + \overline{T}_{\bar{\imath}})$, arising from the determinant of the internal metric. In the string frame, such a volume factor arises by dimensional reduction in the untwisted sector, but is clearly absent in twisted sectors. From a purely low-energy supergravity point of view, in turn, it is possible and in fact convenient to move to the so-called Einstein frame, where in particular the volume dependence in the untwisted sector part of the effective action is reabsorbed through a suitable definition of the metric. In this framework, it is then plausible to expect that target-space anomalies can be computed as simple gauge anomalies. This is indeed the approach adopted in [21]. Clearly, the final result for the total anomaly should be identical in the two approaches, and at most, one expects a reshuffling of the single contributions to the anomaly from those states affected by the frame redefinition. In fact, the comparison between the string-derived anomaly encoded in (3.28)–(3.29) and that expected from an Einstein frame supergravity approach as in [21] can be done, and seems to lead to a discrepancy.

Consider first mixed target-space-gauge and target-space-gravitational anomalies. The relevant components of the polynomial I_{FT} given by the sum of (3.28) and (3.29) are found to be of the form $I_{GFF} = 1/2(2\pi)^3 \sum_{i,a} b_a^i \, G_i \wedge F_a{}^n \wedge F_{an}$ and $I_{GRR} = 1/48(2\pi)^3 \sum_i b^i \, G_i \wedge R^{\mu\nu} \wedge R_{\mu\nu}$, with coefficients b_a^i and b_i given by:

$$\begin{aligned} b_a^i &= -C(G_a) + \sum_{R_a} T(R_a)(1 + 2n^i_{R_a}) \,, \\ b^i &= 21 + 1 + \delta_T - \dim(G) + \sum_\alpha (1 + 2n^i_\alpha) \,. \end{aligned} \tag{4.1}$$

The correct modular weight n_i is automatically obtained for each state. In b_a^i, the first contribution is from the gauge bosons and involves the quadratic Casimir $C(G_a)$ of the relevant group factor G_a, whereas the sum in the second term runs over charged states with non-trivial representations R_a with respect to the gauge group factor G_a, with character $T(G_a)$. In b^i instead, the 21 corresponds to the gravitino, the 1 to the dilatino, the quantity δ_T to the untwisted moduli, and the sum in the last term is over all matter states α. The above expressions for the coefficients b_a^i and b_i coincide term by term with the well-known results of [19, 20], for each single contribution, showing perfect agreement for mixed anomalies

In the case of pure target-space anomalies, instead, the string-derived results extracted from (3.28)–(3.29) and those expected from the supergravity analysis of [21] differ quite radically in the untwisted sector, although agreement is found for twisted sectors. Not only the single contributions from each untwisted state disagree, but they also sum up to different total results, thus showing a true discrepancy.

The anomaly we find therefore differs concretely from the one expected from previous analyses in the literature, beyond the terms linear in target-space curvatures. The difference could be related to subtleties associated to the compositeness of the target-space connection, which could invalidate the analogy with a gauge connection

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when many of them occur as external states. As a matter of fact, it turns out that the total anomaly derived from our string computation nicely factorizes, whereas we have checked that the total anomaly computed by assuming that the target-space dependence is analogous to the gauge dependence does not factorize. We take this as an indirect argument in favour of our results, and therefore assume that they should be reproduced by a more careful supergravity analysis.

The two explicit examples that we have analysed in detail exhibit a certain number of properties that we expect to be quite general, and valid for any orbifold model free of U(1) anomalies and fixed planes. Defining the quantity $\mathrm{tr}\, F^2$ to represent the appropriate sum of terms over the various group factors G_a arising in a given model: $\mathrm{tr}\, F^2 = 1/2 \sum_a F_a{}^n \wedge F_{an}$, the anomaly can be written in a universal way as $I_{FT} = X_2 \wedge X_4$ with:

$$X_2 = \frac{15}{2(2\pi)^3} \left(\sum_{i=1}^{3} G_i \right) , \tag{4.2}$$

$$X_4 = \mathrm{tr}\, R^2 - \mathrm{tr}\, F^2 + 2 \sum_{i=1}^{3} G_i^2 \, . \tag{4.3}$$

These same quantities enter also the GS term $-2\pi B \wedge X_2$ and the modified Bianchi identity $dH = X_4$. These general results encode full information about all possible target-space, gauge and gravitational anomalies in four-dimensional heterotic vacua without anomalous U(1)'s and threshold corrections, as well as the GS mechanism through which they are cancelled. The pure target-space part is a novel result, at both the field-theory and the string-theory level. The generalization to models with an anomalous U(1) does not present any technical problem, and only the two-form X_2 entering the GS term is expected to be modified, through additional terms proportional to the field strength of the anomalous U(1). Models with fixed planes and threshold corrections, instead, do present interesting new features [17].

The details of the GS mechanism of anomaly cancellation occurring at the level of low-energy effective action do not present any qualitative novelty. As usual, the cancellation mechanism can be understood both in a linear and a chiral multiplet description [18, 19]. In the linear basis, the part of the effective lagrangian relevant to anomaly cancellation is given by

$$L_l = -\frac{1}{12} \left| dB - X_4^{(0)} \right|^2 - 2\pi B \wedge X_2 \, . \tag{4.4}$$

The modified kinetic term of B requires that $\delta B = X_4^{(1)}$ under gauge, gravitational or target-space transformations. The GS term induces an anomaly $\delta L_l = -2\pi X_2 \wedge X_4^{(1)}$, which is equivalent to the WZ descent of $I_{GS} = -X_2 \wedge X_4$, modulo irrelevant local terms. This exactly cancels the field-theory anomaly $I_{FT} = X_2 \wedge X_4$. In the chiral

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basis, the relevant lagrangian for the axion field χ is instead:

$$L_c = \frac{1}{2}\left|d\chi - X_2^{(0)}\right|^2 - 2\pi\chi X_4\,. \tag{4.5}$$

The shift in the kinetic term now involves only the target-space connection. This requires $\delta\chi = X_2^{(1)}$ under target-space transformations. The GS term induces an anomaly $\delta L_c = -2\pi X_2^{(1)} \wedge X_4$, which is again equivalent to the WZ descent of $I_{GS} = -X_2 \wedge X_4$, cancelling as before the field-theory anomaly $I_{FT} = X_2 \wedge X_4$. In both approaches, one can easily generalize the above results for the CP-odd part to full superspace expressions, encoding for instance also CP-even target-space anomalies.

5. Geometric structure and relation to torsion

The general structure of the anomalies derived in the previous sections can be understood quite nicely from a geometrical point of view, and reveals some interesting features. More precisely, it is possible to understand both eqs. (4.2) and (4.3) from a ten-dimensional point of view.

Let us begin by recalling some well-known facts about the ten-dimensional case. The tree-level effective lagrangian of the $D = 10$ heterotic theory exhibits a modified kinetic term for the NSNS two-form B, involving the field-strength $H = dB - Y_4^{(0)}$, where the Chern-Simons three-form $Y_4^{(0)}$ is defined as the WZ descent of the four-form

$$Y_4 = \mathrm{tr}\, R^2 - \mathrm{tr}\, F^2\,. \tag{5.1}$$

Correspondingly, the NSNS field has a modified Bianchi identity $dH = Y_4$ and transforms non-trivially as $\delta B = Y_4^{(1)}$ under gauge transformations or diffeomorphisms. The well-known GS mechanism then occurs through the variation of the string-generated one-loop effective coupling $-2\pi B \wedge Y_8$, where the eight-form Y_8 is given by

$$Y_8 = \frac{1}{8}\,\mathrm{tr}\, R^4 + \frac{1}{32}(\mathrm{tr}\, R^2)^2 + \frac{1}{4}(\mathrm{tr}\, F^2)^2 - \frac{1}{8}\,\mathrm{tr}\, F^2\,\mathrm{tr}\, R^2\,. \tag{5.2}$$

An important thing to note is that the two-form R entering both (5.1) and (5.2) is in fact a generalized curvature hiding also a dependence on the torsion H.[9] Indeed, it has been shown in [34] that various requirements, such as space-time supersymmetry and world-sheet conformal invariance, fix unambiguously the connection ω which defines R in the above equations to be the sum of the usual spin connection and the torsion connection constructed from H. This is also clear from a direct computation of the anomaly along the lines of section 3, which is easily generalized to include the effect of torsion through vertex operators for the B field.

Let us now consider an orbifold compactification leading to $D = 4$. The spectrum of the states arising from the untwisted spectrum can easily be deduced starting from

[9] The Bianchi identity for H then must be understood and solved iteratively, in an α' expansion.

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$D = 10$. In particular, the $D = 10$ graviton and NSNS two-form B give rise (in addition to the $D = 4$ graviton, two-form B and possibly other scalars) to the three diagonal untwisted complex fields (3.3). In standard orbifold compactifications, the internal metric and NSNS flux are fixed, and the vacuum expectation value of the T-moduli is constant, implying in particular that there is no background torsion in the internal manifold. However, a non-trivial generalized connection is induced by the fluctuations of the complexified internal metric represented by the untwisted T-moduli. One finds

$$\omega_\mu{}^{i\bar{\imath}} = Z_{\mu i} \,, \tag{5.3}$$

where Z_i are precisely the target-space connections (3.16). Correspondingly, the internal components of the generalized curvature two-form R do not vanish, but rather $R_{i\bar{\imath}} = G_i$, and the ten-dimensional quantity $\mathrm{tr}\, R^2$ decomposes into its four-dimensional analogue, plus a contribution coming from the internal components:

$$\mathrm{tr}_{D=10}\, R^2 = \mathrm{tr}_{D=4}\, R^2 + 2\sum_{i=1}^{3} G_i^2 \,. \tag{5.4}$$

The general structure (4.3) of X_4 in $D = 4$ is hence completely fixed by the dimensional reduction of the well-known expression (5.1) for Y_4 in $D = 10$. A similar interpretation can be given also for X_2 in (4.2). Indeed, the two-form X_2 entering the GS term in a $D = 4$ model can be obtained by integrating the $D = 10$ GS term Y_8 given by (5.2) over the compactification manifold. Much in the same way as U(1) terms in X_2 comes from couplings in (5.2) with one gauge curvature in the four-dimensional directions [23, 24], target-space terms can arise from the torsion dependence in (5.2).

Using the geometric interpretation just described, one can also obtain a much better understanding of the various characteristic classes appearing in the contribution of the various chiral fields to (3.28) and (3.29). Again, as far as untwisted states are concerned, it is convenient and justified to reinterpret the result from a $D = 10$ point of view, in which target-space transformations (3.1) are directly related to internal reparametrizations of the orbifold. The first line in (3.28) clearly comes from a chiral gravitino in $D = 10$, which gives rise, when dimensionally reduced to $D = 4$, to a chiral gravitino transforming as an internal spinor (first term), plus chiral spinors transforming as an internal gravitino (second term), with respect to the reparametrizations of the orbifold induced by (3.1). The second and third lines in (3.28), instead, come from charged $D = 10$ chiral spinors, the gauginos, which give rise to multiplets of chiral spinors in $D = 4$ transforming as internal spinors. Twisted states arise from given fixed points of the orbifold and therefore cannot be understood from a $D = 10$ point of view. It is clear from the form of (3.29) that they are simply $D = 4$ spinors and transform just as U(1) charged fields under reparametrizations of the orbifold, with charges given by the modular weights n_i.

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As already noticed in section 5, the reason for the different structure of anomalies in untwisted and twisted sectors is related to the very different way in which the corresponding states arise geometrically.

6. Conclusions

We have shown that target-space anomalies in heterotic orbifold models can be understood on the same footing as gauge and gravitational anomalies, allowing for a more general and unified analysis of them in such models. All anomalies cancel through a GS mechanism mediated by the dilaton multiplet, thanks to a new term in the Bianchi identity for the NSNS B-field that involves the target-space curvature. The key property behind this cancellation is again modular invariance, and both the quantum anomaly and the classical GS term are encoded in a generalized elliptic genus with well-defined modular properties. Although we have focused on the simplest four-dimensional orbifold models, it is clear that the validity of these results extends straightforwardly to a much wider class of models, involving for example non-standard embedding, Wilson lines, or a different number of space-time dimensions. A similar analysis should also be possible for models with fixed planes and threshold corrections in $N = 2$ sectors.

Several interesting novelties have emerged in the low-energy understanding of target-space symmetry as well. For instance, a precise relation to torsion has emerged, which leads to a nice geometric interpretation. Our results for quantum anomalies involving more than one target-space connection disagree from the naive field theory expectation based on the analogy with a gauge connection, suggesting interesting technical subtleties to be unravelled.

A very interesting application of target-space symmetries has recently emerged in the context of a possible duality between certain heterotic orbifold and type-IIB orientifold $N = 1$ vacua in $D = 4$ [9]–[13]. Whereas these symmetries have a good reason to persist quantum mechanically on the heterotic side (the underlying T-duality), they are apparently accidental on the orientifold side, and therefore constitute a very stringent test of the proposed duality. It has been shown recently in [7] that one-loop anomalies in the simplest $\mathbb{Z}_3$ and $\mathbb{Z}_7$ orientifold models are cancelled through a generalized GS mechanism mediated by several twisted RR fields, as proposed in [3]. Together, the results derived in this work for heterotic models and those derived in [7] for type-IIB orientifold models demonstrate that at least the simplest vacua in both theories do indeed have the same target-space symmetry and are free of any anomaly. In spite of other problems pointed out so far in the literature [4, 6], this is quite a suggestive result in favour of the conjectured duality. Finally, it is interesting to note that the same peculiar structure of the target-space anomalies appears in both heterotic orbifolds and type-IIB orientifolds.

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A. Characteristic classes

The relevant characteristic classes appearing in the anomaly from chiral spinors and Rarita-Schwinger fields are the Roof-genus and the G-polynomial [37], functions of the gravitational curvature R and defined in terms of the skew eigenvalues λ_a of $R/2\pi$ as:

$$\widehat{A}(R) = \prod_{a=1}^{D/2} \frac{\lambda_a/2}{\sinh \lambda_a/2}\,, \tag{A.1}$$

$$\widehat{G}(R) = \prod_{a=1}^{D/2} \frac{\lambda_a/2}{\sinh \lambda_a/2} \left(2\sum_{b=1}^{D/2} \cosh \lambda_b - 1 \right). \tag{A.2}$$

For the target-space dependence, similar characteristic classes turn out to appear; these are functions of the composite curvature $G = dZ$ and defined, in terms of $g_i = G_i/2\pi$, as

$$\widehat{A}_k(G) = \prod_{i=1}^{3} \frac{\sin(\pi k v_i)}{\sin(g_i/2 + \pi k v_i)}\,, \tag{A.3}$$

$$\widehat{G}_k(G) = \prod_{i=1}^{3} \frac{\sin(\pi k v_i)}{\sin(g_i/2 + \pi k v_i)} \left(2\sum_{j=1}^{3} \cos(g_j + 2\pi k v_j) - 1 \right). \tag{A.4}$$

The gauge dependence is influenced by the orbifold projection, which acts on the SU(3) part of the maximal gauge group and projects it to a subgroup H. The element g^k of the orbifold group $\mathbb{Z}_N$ acts as $F/2\pi \to F/2\pi + 2\pi i k V$ on the rescaled gauge curvature of the visible gauge group, where V is an SU(3) shift with skew-eigenvalues v_i. This shift acts as $y_i \to y_i + 2\pi k v_i$ for $i = 1, 2, 3$. It is then natural to define the following twisted Chern characters:

$$\mathrm{ch}^k(F) = \mathrm{ch}(F + (2\pi)^2 k V)\,. \tag{A.5}$$

Finally, it is convenient to define:

$$\mathrm{ch}^k_{q_i}(F) = \exp\left\{ i\sum_{i=1}^{3} q_i (y_i + 2\pi k v_i) \right\}, \tag{A.6}$$

$$\mathrm{ch}^k_{n_i}(G) = \exp\left\{ -i\sum_{i=1}^{3} (1 + 2n_i) \left(\frac{g_i}{2} + \pi k v_i \right) \right\}. \tag{A.7}$$

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B. Decomposition of characters

We report in this appendix some useful explicit expressions for the Chern characters relevant to orbifold models with standard embedding. Recall first that the SO(16) group underlying E_8 decomposes naturally into SO(6) $\times$ SO(10), and the eight skew eigenvalues of the rescaled SO(16) curvature $F/2\pi$ (y_i, $i = 1, \ldots, 8$) split into the three skew eigenvalues of the SO(6) curvature (y_i, $i = 1, \ldots, 3$) and the five skew eigenvalues of the SO(10) curvature (y_i, $i = 4, \ldots, 8$). The internal SO(6) $\sim$ SU(4) group further decomposes into U(1) $\times$ SU(3), the rescaled U(1) curvature being $\sum_i y_i/6$. For standard embeddding, one of the two E_8 groups of the $D = 10$ theory is broken to $E_6 \times H$ in $D = 4$, where the E_6 group is constructed from SO(10) $\times$ U(1) and H is a subgroup of SU(3). The Chern characters of the relevant E_8 and E_6 representations can then be derived using their decomposition with respect to the defining SO(16) and SO(10) subgroups respectively.

The basic fundamental and positive/negative spinor representations of SO($2n$) are easily computed. One finds:

$$\begin{aligned} \mathrm{ch}_{\mathbf{2n}}(F) &= 2\sum_{i=1}^{n} \cos y_i \,, \\ \mathrm{ch}_{\mathbf{2^{n-1}}}(F) &= 2^{n-1}\left(\prod_{i=1}^{n} \cos\frac{y_i}{2} \pm \prod_{i=1}^{n} i \sin\frac{y^i}{2}\right), \end{aligned} \tag{B.1}$$

in terms of the skew-eigenvalues y_i of the field strength $F/2\pi$. The Chern character in the adjoint representation is then obtained as

$$\mathrm{ch}_{\mathbf{n(2n-1)}}(F) = \frac{1}{2}\left[\mathrm{ch}^2_{\mathbf{2n}}(F) - \mathrm{ch}_{\mathbf{2n}}(2F)\right] . \tag{B.2}$$

The adjoint representation **248** of E_8 decomposes as $\mathbf{248} = \mathbf{120} \oplus \mathbf{128}$ under SO(16). Its Chern character is then found to be

$$\mathrm{ch}_{\mathbf{248}}(F) = 8 + 128\left(\prod_{i=1}^{8} \cos\frac{y_i}{2} + \prod_{i=1}^{8} \sin\frac{y_i}{2}\right) + 2\sum_{i\neq j=1}^{8} \cos y_i \cos y_j \,. \tag{B.3}$$

The relevant representations of E_6 and SU(3) are those appearing in the decomposition of the adjoint of E_8 under the maximal subgroup $E_6 \times$ SU(3):

$$\mathbf{248} \to (\mathbf{78}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{8}) \oplus (\mathbf{27}, \mathbf{3}) \oplus (\overline{\mathbf{27}}, \bar{\mathbf{3}}) \,. \tag{B.4}$$

The adjoint **78** and "fundamental" **27** representations of E_6 further decompose as follows with respect to the SO(10) $\times$ U(1):

$$\begin{aligned} \mathbf{78} &\to \mathbf{45}_0 \oplus \mathbf{16}_3 \oplus \overline{\mathbf{16}}_{-3} \oplus \mathbf{1}_0 \,, \\ \mathbf{27} &\to \mathbf{16}_{-1} \oplus \mathbf{10}_2 \oplus \mathbf{1}_{-4} \,. \end{aligned}$$

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The U(1) × SU(3) part of the Chern characters is easily obtained by returning to the original SO(6) representations. For example, using the decomposition of the **4** of SO(6) as $\mathbf{1_3} \oplus \mathbf{3_{-1}}$ under U(1) × SU(3) and applying (B.1), one can obtain the Chern characters in the $\mathbf{1_3}$ and the $\mathbf{3_{-1}}$ representations. Generalizing to an arbitrary U(1) charge, one finds

$$\mathrm{ch}_{\mathbf{1_q}}(F) = \exp\left(iq\sum_{i=1}^{3}\frac{y_i}{6}\right),$$
$$\mathrm{ch}_{\mathbf{3_q}}(F) = \exp\left(i(q-2)\sum_{i=1}^{3}\frac{y_i}{6}\right)\sum_{i=1}^{3}\exp iy_i\,.$$

Similarly, the Chern character of the adjoint representation **8** of SU(3) is easily deduced from the decomposition $\mathbf{3}\otimes\bar{\mathbf{3}} = \mathbf{8}\oplus\mathbf{1}$. The explicit form of the Chern characters for the remaining relevant representations is then found to be:

$$\mathrm{ch}_{(\mathbf{78},\mathbf{1})}(F) = 6 + 32\left[\cos\left(\sum_{i=1}^{3}\frac{y_i}{2}\right)\prod_{j=4}^{8}\cos\frac{y_j}{2} + \sin\left(\sum_{i=1}^{3}\frac{y_i}{2}\right)\prod_{j=4}^{8}\sin\frac{y_j}{2}\right] + \\ + 2\sum_{j\neq k=4}^{8}\cos y_j\cos y_k\,, \tag{B.5}$$

$$\mathrm{ch}_{(\mathbf{1},\mathbf{8})}(F) = 2 + \sum_{i\neq j=1}^{3}\cos(y_i - y_j)\,, \tag{B.6}$$

$$\mathrm{ch}_{(\mathbf{27},\mathbf{3})}(F) = \left[\exp\left(-i\sum_{i=1}^{3}y_i\right) + 16\exp\left(-i\sum_{i=1}^{3}\frac{y_i}{2}\right)\left(\prod_{j=4}^{8}\cos\frac{y_j}{2} + \prod_{j=4}^{8}\sin\frac{y_j}{2}\right) + \right. \\ \left. + 2\sum_{j=4}^{8}\cos y_j\right]\sum_{i=1}^{3}\exp iy_i\,. \tag{B.7}$$

Notice finally that the decomposition (B.4) can be technically understood on the Chern characters (B.3), (B.5)–(B.7) by using the following trigonometric identity

$$\prod_{i=1}^{3}2\sin\frac{y_i}{2} = \sum_{i=1}^{3}2\sin\left(\sum_{j=1}^{3}\frac{y_j}{2} - y_i\right) - 2\sin\left(\sum_{i=1}^{3}\frac{y_i}{2}\right),$$
$$\prod_{i=1}^{3}2\cos\frac{y_i}{2} = \sum_{i=1}^{3}2\cos\left(\sum_{j=1}^{3}\frac{y_j}{2} - y_i\right) + 2\cos\left(\sum_{i=1}^{3}\frac{y_i}{2}\right),$$

implementing the SO(6) → U(1) × SU(3) decomposition.

C. Limits of the partition functions

In this appendix, we report some useful details about the computation of the $\tau \to i\,\infty$ limit of the partition functions $Z_{ST}(R)$, $Z_C^{k,l}(G)$ and $Z_\Gamma^{k,l}(F)$ in eqs. (3.19)–(3.21).

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In the untwisted sector, the limits are easily obtained, and one finds:

$$Z_{ST}(R) \to q^{-1/12}\left[\widehat{A}(R) + \left(\widehat{G}(R) - \widehat{A}(R)\right) q\right], \tag{C.1}$$

$$Z_C^{k,l=0}(G) \to \epsilon_k N_k^{-1/2}\, q^{-1/4}\left[1 + \left(\widehat{A}_k(G) + \widehat{G}_k(G)\right) q\right], \tag{C.2}$$

$$\begin{aligned} Z_\Gamma^{k,l=0}(F) \to q^{-2/3}\Big[1 + \Big(&\mathrm{ch}_{\mathbf{248}}(F') + \mathrm{ch}_{(\mathbf{78},\mathbf{1})}(F) + \mathrm{ch}^k_{(\mathbf{1},\mathbf{8})}(F) + \\ &+ \mathrm{ch}^k_{(\mathbf{27},\mathbf{3})}(F) + \mathrm{ch}^k_{(\overline{\mathbf{27}},\overline{\mathbf{3}})}(F)\Big) q\Big]. \end{aligned} \tag{C.3}$$

Combining these expressions, the result (3.28) is obtained.

In the twisted sector, the situation is more involved. Indeed, there are in general matter states with both vanishing and non-vanishing oscillator number N_L; correspondingly, there will be a complicated interference of terms between Z_C and Z_Γ in the limit $\tau \to i\,\infty$. For this reason, it is convenient to recall how these states arise in general. The mass-shell condition for a generic massless state in the l-th twisted sector reads

$$N_L - 1 + \frac{\eta}{2} + \frac{(p+vl)^2}{2} = 0\,, \tag{C.4}$$

where $\eta = \sum_{i=1}^3 \theta_i^l(1-\theta_i^l)$ represents the shift in the zero-point energy in each model and takes the value $2/3$ for $\mathbb{Z}_3$ and $4/7$ for $\mathbb{Z}_7$. Such a state is in general created from the Fock vacuum, with certain numbers m_i and $\tilde{m}_i$ of left- and right-moving bosonic creation operators in the i-th internal direction, determining an oscillator number N_L; it is associated with lattice vectors p satisfying $(p+v)^2 = 2-2N_L-\eta$. Since this is the most general massless twisted state, the relevant terms to keep in Z_{ST}, Z_C and Z_Γ are those of the type $Z_{ST} \sim \bar{q}^{-\frac{1}{12}}$, $Z_C \sim \bar{q}^{-\frac{1}{4}+\frac{\eta}{2}+N_L}$ and $Z_\Gamma \sim \bar{q}^{-\frac{2}{3}+\frac{(p+v)^2}{2}}$. Indeed, the corresponding term in the total partition function is then $Z \sim q^{N_L-1+\frac{\eta}{2}+\frac{(p+v)^2}{2}}$, and represents a massless contribution whenever the mass-shell condition (C.4) is satisfied. One can therefore restrict from the beginning to these types of terms when evaluating the partition functions for $\tau \to i\,\infty$.

The limit of the internal partition function Z_C is easily obtained by using the infinite sum representation of θ-functions. The integer n labelling the infinite sum in the definition of the θ-function associated to a given complex direction is related to the level of the bosonic oscillator that creates the corresponding state, and positive/negative values of n correspond to left-/right- moving oscillators. Since all the massless states occurring in the considered models involve at most oscillators of lowest level, it is enough to keep the first two subleading terms (in q) beside the leading term, in each θ-function. One then finds:

$$Z_C^{k,l\neq 0}(G) \to -i\,\epsilon_l\left[\bar{q}^{-\frac{1}{4}+\frac{\eta}{2}}\ \mathrm{ch}^k_{-(1-\theta_i^l)}(G) + \sum_s\ \bar{q}^{-\frac{1}{4}+\frac{\eta}{2}+N_L(s)}\,\mathrm{ch}^k_{n_i(s)}(G)\right], \tag{C.5}$$

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with the following expressions for the oscillator number N_L and the modular weight n_i for the generic massless twisted state s:

$$N_L(s) = -\frac{1}{2}\sum_{i=1}^{3}\left[(m_i-\tilde{m}_i)(1-2\theta_i^l)-(m_i+\tilde{m}_i)\right], \tag{C.6}$$

$$n_i(s) = -(1-\theta_i^l+m_i-\tilde{m}_i)\,. \tag{C.7}$$

Notice that the known expression [20] for the modular weight of a generic oscillator state is recovered.

For the lattice partition function, one has to understand which kind of lattice vectors p can occur for the generic state discussed above, and which representation is defined by the corresponding $q = p + v$. For the $N_L = 0$ states, the allowed p's give rise to a **27** of E_6 and fixed charges in the SU(3) part, for both the $\mathbb{Z}_3$ and $\mathbb{Z}_7$ models. For the states with $N_L \neq 0$, the allowed p's lead to three distinct q's in the SU(3) part, which we shall label with q^a, $a = 1, 2, 3$, giving rise to a triplet for $\mathbb{Z}_3$ and three groups of eight singlets for $\mathbb{Z}_7$. More precisely, the relevant part of the lattice partition function is found to be:

$$Z_\Gamma^{k,l\neq 0}(F) \to \bar{q}^{\frac{1}{3}-\frac{\eta}{2}}\ \mathrm{ch}_{(\mathbf{27},\mathbf{1})}(F)\ \mathrm{ch}^k_{\theta_i^l+\frac{\epsilon_l-3}{6}}(F) + \sum_{a=1}^{3}\bar{q}^{-\frac{2}{3}+(q^a)^2}\ \mathrm{ch}^k_{(q^a)_i}(F)\,. \tag{C.8}$$

One can check that the allowed $q(s)$'s for an oscillator state s, among the three q^a, $a = 1, 2, 3$, are defined by

$$q_i^a(s) = \theta_i^l - \epsilon_l\delta_i^a + \frac{\epsilon_l - 1}{2}\,, \qquad a \in \left\{a|\epsilon_l\theta_a(l) + \frac{1-\epsilon_l}{2} = N_L(s)\right\}. \tag{C.9}$$

Putting everything together, one finally finds

$$I_{FT}^{\mathrm{tw.}}(F,R,G) = -\frac{i}{2N}\sum_{k=0}^{N-1}\sum_{l=1}^{N-1}N_{k,l}\,\epsilon_l\left[\mathrm{ch}_{(\mathbf{27},\mathbf{1})}(F)\ \mathrm{ch}^k_{\theta_i^l+\frac{\epsilon_l-3}{6}}(F)\ \mathrm{ch}^k_{-(1-\theta_i^l)}(G) + \right.$$
$$\left. + \sum_s \mathrm{ch}^k_{q_i(s)}(F)\ \mathrm{ch}^k_{n_i(s)}(G)\right]\widehat{A}(R)\,, \tag{C.10}$$

where the sum over the set of oscillator states s now contains also the sum over the allowed q's. The charges $q_i(s)$ and modular weights $n_i(s)$ are given by (C.7) and (C.9) respectively. Observe finally that to each particle s in the l-th twisted sector there corresponds an antiparticle s' in the $(N-l)$-th conjugate sector with left- and right-moving oscillators exchanged and opposite charge vector. Since $N_L(s') = N_L(s)$, $n_i(s') = 1 - n_i(s)$ and $q_i(s') = -q_i(s)$, these two states s and s' give identical contributions to the anomaly, as expected. Pairing these contributions, the sums over l can be reduced to $l \in S_l$. The sum over k can be done explicitly by noticing that the k-dependence hidden in the twisted Chern characters amounts to the phase $\exp 2\pi k i \sum_i v_i(q_i - n_i)$, which can be checked case by case to be simply 1. One then obtains eq. (3.29).

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D. Chiral determinants

In this appendix, we provide some details about the evaluation of the determinants appearing in section 3. In particular, we show how two different regularizations — modular-invariant and holomorphic — give different results: the elliptic genera $\bar{A}(\tau; F, R, G)$ and $A(\tau; F, R, G)$ respectively. The typical quantity to be computed is a chiral determinant with periodicities $\alpha, \beta \in [0, 1[$ and a twist λ:[10]

$$\det_{\alpha,\beta}(\lambda) = \prod_{m,n} \frac{2\pi}{\tau_2} \left[(m - \beta) - (n + \alpha)\, \tau + \lambda \right]. \tag{D.1}$$

It is well known that chiral determinants such as (D.1) are ambiguous. A possible approach is to multiply (D.1) by its complex conjugate, which corresponds to the opposite chirality. However, although the square modulus of (D.1) can be computed, it fails to factorize into the product of a holomorphic and an anti-holomorphic parts associated to components with opposite chirality. In order to define in an unambiguous way the original chiral determinant, one can start with λ and $\bar{\lambda}$ independent of each other, i.e. different twists in the chiral and antichiral parts, and define the original chiral determinant as

$$\det_{\alpha,\beta}(\lambda) \equiv \left. \frac{|\det_{\alpha,\beta}(\lambda)|^2}{\det_{\alpha,\beta}(0)} \right|_{\bar{\lambda}=0}. \tag{D.2}$$

It is worth observing that this definition precisely corresponds to the original form of the determinants yielding the topological generating functional $\bar{A}$. In each space-time direction there is a bosonic determinant with left twist $\lambda \neq 0$ and right twist $\bar{\lambda} = 0$, and a fermionic determinant with $\lambda = 0$.

In the non-degenerate case $(\alpha, \beta) \neq (0, 0)$, the square modulus of (D.1) can be computed from the representation [38]:

$$\begin{aligned} \ln |\det_{\alpha,\beta}(\lambda)|^2 = \frac{1}{2} \lim_{s \to 0} \frac{d}{ds} \sum_n \Bigg[& \int_{C_+} dz \left(\frac{e^{i\pi z}}{i \sin \pi z} f^{-s}_{\alpha,\beta}(z, \lambda) - f^{-s}_{\alpha,\beta}(z, \lambda) \right) + \\ & + \int_{C_-} dz \left(\frac{e^{-i\pi z}}{i \sin \pi z} f^{-s}_{\alpha,\beta}(z, \lambda) + f^{-s}_{\alpha,\beta}(z, \lambda) \right) \Bigg], \end{aligned} \tag{D.3}$$

where

$$f_{\alpha,\beta}(z, \lambda) = \left(\frac{2\pi}{\tau_2} \right)^2 [(z - \beta) - (n + \alpha)\, \tau + \lambda][(z - \beta) - (n + \alpha)\, \bar{\tau} + \bar{\lambda}]. \tag{D.4}$$

We have performed a Sommerfeld-Watson transformation to convert the sum over m to an integral over closed contours $C_\pm$ following the real axis and going at infinity

[10] In our case, the twist λ is always a given two-form, so the expressions that will follow are formal and eventually have to be understood as an expansion in λ.

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in the upper/lower half-plane. The first and third terms in the square brackets converge for $s \to 0$, so that the derivative at $s = 0$ can be taken directly. This gives rise to $\ln f_{\alpha,\beta}(z,\lambda)$, which has branch points in the complex plane at the locations $z_+ = \beta - \lambda + (n+\alpha)\,\tau$ and $z_- = \beta - \bar{\lambda} + (n+\alpha)\,\bar{\tau}$. The corresponding integrals can then easily be performed, taking care of deforming the $C_\pm$ contours to avoid the branch points. Their contribution to $\ln|\det_{\alpha,\beta}(\lambda)|^2$ is

$$\sum_{n=0}^{\infty} \ln\left|1 - q^{n+\alpha} e^{2\pi i(\beta-\lambda)}\right|^2 + \sum_{n=1}^{\infty} \ln\left|1 - q^{n-\alpha} e^{-2\pi i(\beta-\lambda)}\right|^2 . \tag{D.5}$$

Let us now consider the second and fourth terms in the square brackets in (D.3). These terms are convergent only for $\mathrm{Re}\, s > 1/2$, so we compute the integrals in this regime and then continue analytically the result to $s = 0$. In this case

$$\begin{aligned} \int_{C_+} dz\, f_{\alpha,\beta}^{-s}(z,\lambda) - \int_{C_-} dz\, f_{\alpha,\beta}^{-s}(z,\lambda) &= 2\int_{-\infty}^{+\infty} dx\, f_{\alpha,\beta}^{-s}(x,\lambda) \\ &= 2[(n+\alpha)\tau_2 - \mathrm{Im}\,\lambda]^{1-2s} \frac{\Gamma(1/2)\Gamma(s-1/2)}{\Gamma(s)} . \end{aligned} \tag{D.6}$$

The sum over n can then be performed using the ζ-function regularization. In this way, one finds a contribution to $\ln|\det_{\alpha,\beta}(\lambda)|^2$ equal to

$$-2\pi\,\tau_2\, B_2\left(\alpha - \mathrm{Im}\,\frac{\lambda}{\tau_2}\right), \tag{D.7}$$

where $B_2(c) = 1/6 + c^2 - c$ is the second Bernoulli polynomial.[11] By combining (D.7) and (D.5), one finally obtains

$$\begin{aligned} |\det_{\alpha,\beta}(\lambda)|^2 = {} & e^{-\frac{2\pi(\mathrm{Im}\,\lambda)^2}{\tau_2}}\, e^{-2\pi\,\mathrm{Im}\,\lambda(1-2\alpha)}\, |q|^{1/6+\alpha^2-\alpha} \times \\ & \times \prod_{n=0}^{\infty} \left|1 - q^{n+\alpha} e^{2\pi i(\beta-\lambda)}\right|^2 \prod_{n=1}^{\infty} \left|1 - q^{n-\alpha} e^{-2\pi i(\beta-\lambda)}\right|^2 . \end{aligned}$$

As anticipated, this expression fails to factorize into holomorphic and and anti-holomorphic parts, but the definition (D.2) yields

$$\det_{\alpha,\beta}(\lambda) = \frac{\theta_1\left[{}^{\alpha}_{\beta}\right](\lambda|\tau)}{\eta(\tau)}\, e^{\frac{\pi\lambda^2}{2\tau_2}} . \tag{D.8}$$

In the degenerate case $(\alpha,\beta) = (0,0)$, the analysis is similar, but one has to introduce an IR regulator [38]. Following the same steps as before, and excluding the zero-mode, one then finds

$$|\det{}'_{0,0}(\lambda)|^2 = (2\pi\lambda)^{-2} e^{-\frac{2\pi\,\mathrm{Im}\,\lambda^2}{\tau_2}}\, e^{-2\pi\,\mathrm{Im}\,\lambda}\, |q|^{1/6} \prod_{n=0}^{\infty} \left|1 - q^n e^{-2\pi i\lambda}\right|^2 \prod_{n=1}^{\infty} \left|1 - q^n e^{2\pi i\lambda}\right|^2 ,$$

[11] Strictly speaking, the result (D.7) is valid only for twists $\alpha < 1/2$ (for sufficiently small λ). However, by considering the equivalent $(1-\alpha)$-twist the result (D.7) can actually be extended to all twists $0 < \alpha < 1$.

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and hence

$$\det{}'_{0,0}(\lambda) = \frac{\theta_1(\lambda|\tau)}{2\pi\lambda\,\eta(\tau)} e^{\frac{\pi\lambda^2}{2\tau_2}} \,. \tag{D.9}$$

Note that this result for the degenerate case can be verified by an alternative method, which consists in rewriting the determinant in terms of Eisenstein series, for which there is a well-known modular-invariant regularization (see for example [27]).

The above results for the determinants give rise to the modular-invariant generalized elliptic genus $\bar{A}(\tau; F, R, G)$, satisfying (3.23) and (3.24). On the other hand, a holomorphic regularization can be easily obtained, for example, by defining the effect of the twist λ by analytic contribution from the twists α and β. In other words, the twisted determinant is defined as the value of the ordinary determinant for twist $(\alpha', \beta') = (\alpha, \beta + \lambda)$. Proceeding in this way, one simply finds the same results as before, (D.8) and (D.9), but without the non-holomorphic exponential term. Without this term, modular invariance is lost, but the results are manifestly holomorphic in τ. The resulting elliptic genus is then $A(\tau; F, R, G)$.

The impossibility of having at the same time holomorphicity and modular invariance is called Quillen's anomaly [39]; it plays a crucial role in anomaly cancellation in heterotic theories.

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